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# The effect of radial acceleration on the electric and magnetic fields of circular currents and rotating charges

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**Abstract**
It is shown that time-independent circular currents and uniformly rotating charge distributions create heretofore unreported constant electric and magnetic fields associated with radial acceleration of the charges forming the circular currents and with radial acceleration of the charges comprising the rotating charge distributions. These fields are computed for several types of rotating charge distributions and for several types of circular currents. One of the consequences of the existence of these fields is that the Aharonov–Bohm effect can now be explained on the basis of classical electrodynamics.

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## 1. Introduction

It is well known that the electric field of an electric charge in the state of uniform translational motion is different from the electric field of the same charge at rest. Therefore it is reasonable to expect that the electric field of a charge in the state of uniform rotational motion should also be different from the electric field of the same stationary charge. The theoretical analysis presented in this paper shows that uniformly rotating charge distributions do indeed produce constant electric fields different from those of the same stationary charge distributions. This analysis is based on the 'causal' solutions of Maxwell's equations representing the electric field $\boldsymbol{E}$ and the magnetic flux density field $\boldsymbol{B}$ in a vacuum in terms of their causative sources—electric charges (charge density $\rho$) and electric currents (current density $\boldsymbol{J}$) [1–6, 7 pp 514–6]:

$$\boldsymbol{E} = \frac{1}{4\pi\varepsilon_0}\int\left\{\frac{[\rho]}{r^3} + \frac{1}{r^2 c}\left[\frac{\partial\rho}{\partial t}\right]\right\}\boldsymbol{r}\,\mathrm{d}V - \frac{1}{4\pi\varepsilon_0 c^2}\int\frac{1}{r}\left[\frac{\partial\boldsymbol{J}}{\partial t}\right]\mathrm{d}V \qquad (1)$$

 

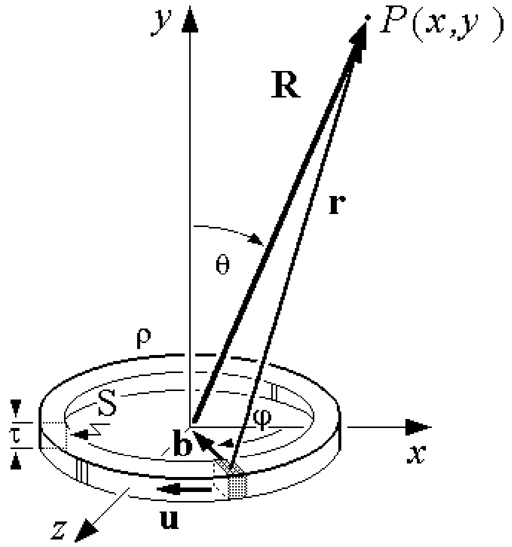


**Figure 1.** A ring carrying a uniformly distributed electric charge of density $\rho$ rotates with constant linear velocity $u$ about its symmetry axis ($y$ axis). The point of observation $P(x, y)$ is in the $xy$ plane.

and

$$\boldsymbol{B} = \frac{\mu_0}{4\pi}\int \frac{[\boldsymbol{J}]}{r^3} \times \boldsymbol{r}\,\mathrm{d}V + \frac{\mu_0}{4\pi c}\int \frac{1}{r^2}\left[\frac{\partial \boldsymbol{J}}{\partial t}\right] \times \boldsymbol{r}\,\mathrm{d}V. \tag{2}$$

In these integrals the brackets are the 'retardation symbol', indicating that the quantities between the brackets are to be evaluated for the 'retarded' time $t' = t - r/c$, where $t$ is the time for which $\boldsymbol{E}$ and $\boldsymbol{B}$ are computed, $r$ is the distance from the source point (volume element $\mathrm{d}V$) to the field point (the point for which $\boldsymbol{E}$ and $\boldsymbol{B}$ are computed), $c$ is the velocity of light, $\varepsilon_0$ is the permittivity of space and $\mu_0$ is the permeability of space. Equations (1) and (2) are of a relatively recent origin and provide a new means for analysing and discussing electric and magnetic phenomena.

Particularly significant for the calculations that follow is the last integral in equation (1). It represents the so-called 'electrokinetic field' $\boldsymbol{E}_\mathrm{k}$ [8], so named because it arises from the motion of electric charges constituting the current density $\boldsymbol{J}$. Although this integral makes an important contribution to the electric field of uniformly moving charges [9, 10], its main function is to represent the effect of charge acceleration on the electric field of the charge (or current) under consideration. In the calculations that follow, we shall use this integral for determining the electric field of electric charge distributions in the state of circular and rotational motion. This type of motion is only possible if the individual charges comprising the charge distribution under consideration experience a radial acceleration. Therefore, even when the speed of the charges is constant, the derivative $\partial\boldsymbol{J}/\partial t$ does not vanish, and an electrokinetic field $\boldsymbol{E}_\mathrm{k}$ is inevitably created.

The derivative $\partial\boldsymbol{J}/\partial t$ appearing in equation (2) indicates that the acceleration-related electrokinetic fields of rotating charges and circular currents are accompanied by acceleration-related magnetic fields. These fields will also be considered in this paper.

## 2. Theory I. Electric field

Consider a charged ring of radius $b$ and cross-sectional area $S$ carrying a uniformly distributed charge of density $\rho$ and rotating about its symmetry axis ($y$ axis) with constant linear velocity $u$ (figure 1). Let us find the electric field produced by this ring at a point $P(x, y)$ at a distance $R$ from the centre of the ring.

According to equation (1), there are two different contributions to the field at $P$. The first contribution comes from the first integral in equation (1). Since the charge density $\rho$ of the ring does not depend on time, the derivative $\partial\rho/\partial t$ vanishes. Furthermore, since the charge density is the same at all times at all points of the ring, the retardation may be assumed to have no effect, so that the first integral in equation (1) represents the ordinary Coulomb field of the ring:

$$\boldsymbol{E}_{\mathrm{C}} = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho}{r^3}\boldsymbol{r}\,\mathrm{d}V. \tag{3}$$

The second contribution to the electric field is from the last integral in equation (1), which represents the electrokinetic field $\boldsymbol{E}_{\mathrm{k}}$. Also in this integral the retardation may be assumed to have no effect because at any given point of the ring the derivative $\partial\boldsymbol{J}/\partial t$ is the same at all times, so that $\partial\boldsymbol{J}/\partial t$ at $t' = t - r/c$ is the same as $\partial\boldsymbol{J}/\partial t$ at $t$. Thus the electrokinetic field created by the ring is

$$\boldsymbol{E}_{\mathrm{k}} = -\frac{1}{4\pi\varepsilon_0 c^2}\int \frac{1}{r}\frac{\partial\boldsymbol{J}}{\partial t}\,\mathrm{d}V. \tag{4}$$

Only $\boldsymbol{E}_{\mathrm{k}}$ depends on the acceleration of the charges, and therefore in the calculations that follow we shall be primarily interested in $\boldsymbol{E}_{\mathrm{k}}$.

Writing the three position vectors shown in figure 1 as

$$\boldsymbol{R} = x\boldsymbol{i} + y\boldsymbol{j} \tag{5}$$

$$\boldsymbol{b} = -b\cos\varphi\boldsymbol{i} - b\sin\varphi\boldsymbol{k} \tag{6}$$

and

$$\boldsymbol{r} = \boldsymbol{b} + \boldsymbol{R} \tag{7}$$

we have

$$\begin{aligned}\boldsymbol{r} &= -b\cos\varphi\boldsymbol{i} - b\sin\varphi\boldsymbol{k} + x\boldsymbol{i} + y\boldsymbol{j}\\ &= (x - b\cos\varphi)\boldsymbol{i} - b\sin\varphi\boldsymbol{k} + y\boldsymbol{j}\end{aligned} \tag{8}$$

where $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ are unit vectors in the direction of the $x$, $y$ and $z$ axis, respectively. The magnitude of $\boldsymbol{r}$ is

$$\begin{aligned}r &= (\boldsymbol{r}\cdot\boldsymbol{r})^{1/2} = [(x - b\cos\varphi)^2 + b^2\sin^2\varphi + y^2]^{1/2}\\ &= (x^2 - 2bx\cos\varphi + b^2 + y^2)^{1/2}.\end{aligned} \tag{9}$$

The derivative $\partial\boldsymbol{J}/\partial t$ for the shaded element of the ring shown in figure 1 is

$$\frac{\partial\boldsymbol{J}}{\partial t} = \frac{\partial\rho\boldsymbol{u}}{\partial t} = \rho\frac{\partial\boldsymbol{u}}{\partial t} = \rho\boldsymbol{a} \tag{10}$$

where $\boldsymbol{a}$ is the radial (centripetal) acceleration of the element[1]. Since

$$\boldsymbol{a} = \frac{u^2}{b^2}\boldsymbol{b} = \frac{u^2}{b^2}(-b\cos\varphi\boldsymbol{i} - b\sin\varphi\boldsymbol{k}) \tag{11}$$

[1] The reason that, for the charge element under consideration, $\partial\boldsymbol{u}/\partial t = \boldsymbol{a}$ is as follows. The partial derivative $\partial\boldsymbol{u}/\partial t$ is the rate of change of $\boldsymbol{u}$ at a *fixed* point of the trajectory of the charge element. At the moment when the charge element arrives at a particular point of its trajectory, its velocity vector is in one direction. When, after a time interval $\Delta t$, the charge element leaves this point, the velocity vector is in a slightly different direction. The change in the velocity vector is $\Delta\boldsymbol{u}$. The partial derivative is the limit of the ratio $\Delta\boldsymbol{u}/\Delta t$ for $\Delta t \to 0$ (very short charge element). The acceleration $\boldsymbol{a}$ is the limit of the same ratio.

we have

$$\frac{\partial \boldsymbol{J}}{\partial t} = -\rho \frac{u^2}{b}(\cos\varphi \boldsymbol{i} + \sin\varphi \boldsymbol{k}). \tag{12}$$

Substituting equations (9) and (12) into (4), we find the contribution of the shaded element shown in figure 1 to the electrokinetic field $\boldsymbol{E}_{\mathrm{k}}$ observed at the point $P(x, y)$:

$$\mathrm{d}\boldsymbol{E}_{\mathrm{k}} = \frac{\rho u^2(\cos\varphi \boldsymbol{i} + \sin\varphi \boldsymbol{k})}{4\pi\varepsilon_0 c^2 b(x^2 - 2bx\cos\varphi + b^2 + y^2)^{1/2}}\,\mathrm{d}V \tag{13}$$

where $\mathrm{d}V$ is the volume of the shaded element. The electrokinetic field produced by the entire ring is therefore

$$\boldsymbol{E}_{\mathrm{k}} = \frac{\rho u^2}{4\pi\varepsilon_0 bc^2}\int \frac{(\cos\varphi \boldsymbol{i} + \sin\varphi \boldsymbol{k})}{(x^2 - 2bx\cos\varphi + b^2 + y^2)^{1/2}}\,\mathrm{d}V \tag{14}$$

where the integration is over the volume of the ring. Expressing $\mathrm{d}V$ as $\mathrm{d}V = Sb\,\mathrm{d}\varphi$, we have

$$\boldsymbol{E}_{\mathrm{k}} = \frac{\rho u^2 S}{4\pi\varepsilon_0 c^2}\int_0^{2\pi} \frac{(\cos\varphi \boldsymbol{i} + \sin\varphi \boldsymbol{k})}{(x^2 - 2bx\cos\varphi + b^2 + y^2)^{1/2}}\,\mathrm{d}\varphi. \tag{15}$$

By the symmetry of the system, the $z$ component of $\boldsymbol{E}_{\mathrm{k}}$ vanishes, so that we are left with

$$\boldsymbol{E}_{\mathrm{k}} = \boldsymbol{i}\frac{\rho u^2 S}{4\pi\varepsilon_0 c^2}\int_0^{2\pi} \frac{\cos\varphi}{(x^2 - 2bx\cos\varphi + b^2 + y^2)^{1/2}}\,\mathrm{d}\varphi. \tag{16}$$

A closed form solution of the integral in equation (16) does not exist. Therefore we shall apply equation (16) to several special cases for which an approximate solution of equation (16) can be obtained.

### 2.1. A small rotating ring

The first special case that we shall consider is a ring as in figure 1 except that its radius is much smaller than the distance between the ring and the point of observation, $b \ll R$. In this case we can write

$$\begin{aligned} 1/(x^2 - 2bx\cos\varphi + b^2 + y^2)^{1/2} &\approx 1/\{(x^2 + y^2)^{1/2}[1 - 2bx\cos\varphi/(x^2 + y^2)]^{1/2}\} \\ &= 1/[R(1 - 2bx\cos\varphi/R^2)^{1/2}] \approx \frac{1 + bx\cos\varphi/R^2}{R} \end{aligned} \tag{17}$$

which, by equation (16), gives

$$\boldsymbol{E}_{\mathrm{k}} \approx \boldsymbol{i}\frac{\rho u^2 S}{4\pi\varepsilon_0 c^2 R}\int_0^{2\pi} \cos\varphi\left(1 + \frac{bx\cos\varphi}{R^2}\right)\mathrm{d}\varphi \tag{18}$$

and, after integration,

$$\boldsymbol{E}_{\mathrm{k}} \approx \boldsymbol{i}\frac{\rho u^2 bxS}{4\varepsilon_0 R^3 c^2} = \boldsymbol{i}\frac{\rho u^2 bS\sin\theta}{4\varepsilon_0 R^2 c^2} \tag{19}$$

or, in terms of spherical coordinates $R$ and $\theta$,

$$\boldsymbol{E}_{\mathrm{k}} \approx \frac{\rho u^2 bS}{4\varepsilon_0 R^2 c^2}(\sin\theta \boldsymbol{R}_u + \sin\theta\cos\theta \boldsymbol{\Theta}_u) \tag{20}$$

where $\boldsymbol{R}_u$ and $\boldsymbol{\Theta}_u$ are unit vectors in the directions of $R$ and $\theta$, respectively.

Using the same approximation as in equation (17), we have for $\boldsymbol{r}/r^3$ in equation (3)

$$\frac{\boldsymbol{r}}{r^3} \approx \frac{1 + 3bx\cos\varphi/R^2}{R^3}\boldsymbol{R} \tag{21}$$

which gives for the Coulomb field of the ring

$$\boldsymbol{E}_{\mathrm{C}} \approx \frac{\rho b S}{2\varepsilon_0 R^3}\boldsymbol{R} = \frac{\rho b S}{2\varepsilon_0 R^2}\boldsymbol{R}_u. \tag{22}$$

The total electric field of the ring observed in the $xy$ plane is obtained by adding equations (20) and (22):

$$\boldsymbol{E} \approx \frac{\rho b S}{2\varepsilon_0 R^2}\left(1 + \frac{u^2}{2c^2}\sin^2\theta\right)\boldsymbol{R}_u + \frac{\rho b S u^2}{8\varepsilon_0 R^2 c^2}\sin(2\theta)\boldsymbol{\Theta}_u. \tag{23}$$

Expressing $\boldsymbol{E}$ in terms of the charge of the ring, $q = 2\pi b S \rho$, we have

$$\boldsymbol{E} \approx \frac{q}{4\pi\varepsilon_0 R^2}\left(1 + \frac{u^2}{2c^2}\sin^2\theta\right)\boldsymbol{R}_u + \frac{q u^2}{16\pi\varepsilon_0 R^2 c^2}\sin(2\theta)\boldsymbol{\Theta}_u. \tag{24}$$

### 2.2. *A small rotating disc*

Expressing in equation (20) the linear velocity $u$ in terms of the angular velocity $\omega$, replacing $b$ by $x'$ and $S$ by $\tau\,\mathrm{d}x'$, and considering the ring to be a differential element of a disc, we can write

$$\mathrm{d}\boldsymbol{E}_{\mathrm{k}} \approx \frac{\rho\omega^2 x'^3}{4\varepsilon_0 R^2 c^2}(\sin^2\theta\boldsymbol{R}_u + \sin\theta\cos\theta\boldsymbol{\Theta}_u)\tau\,\mathrm{d}x' \tag{25}$$

where $\tau$ is the thickness of the disc (the same as that of the ring). Integrating equation (25) between 0 and $b$, we then obtain for the electrokinetic field of a disc of radius $b \ll R$

$$\boldsymbol{E}_{\mathrm{k}} \approx \frac{\rho\omega^2 b^4 \tau}{16\varepsilon_0 R^2 c^2}(\sin^2\theta\boldsymbol{R}_u + \sin\theta\cos\theta\boldsymbol{\Theta}_u). \tag{26}$$

For the Coulomb field we similarly obtain

$$\boldsymbol{E}_{\mathrm{C}} \approx \frac{\rho b^2 \tau}{4\varepsilon_0 R^2}\boldsymbol{R}_u. \tag{27}$$

Adding equations (26) and (27), we obtain for the total electric field of the rotating disc

$$\boldsymbol{E} \approx \frac{\rho b^2 \tau}{4\varepsilon_0 R^2}\left(1 + \frac{\omega^2 b^2}{4c^2}\sin^2\theta\right)\boldsymbol{R}_u + \frac{\rho\omega^2 b^4 \tau}{32\varepsilon_0 R^2 c^2}\sin(2\theta)\boldsymbol{\Theta}_u. \tag{28}$$

Expressing $\boldsymbol{E}$ in terms of the charge of the disc, $q = \pi b^2 \tau \rho$, we have

$$\begin{aligned}\boldsymbol{E} &\approx \frac{q}{4\pi\varepsilon_0 R^2}\left(1 + \frac{\omega^2 b^2}{4c^2}\sin^2\theta\right)\boldsymbol{R}_u + \frac{q\omega^2 b^2}{32\pi\varepsilon_0 R^2 c^2}\sin(2\theta)\boldsymbol{\Theta}_u \\ &= \frac{q}{4\pi\varepsilon_0 R^2}\left(1 + \frac{u^2}{4c^2}\sin^2\theta\right)\boldsymbol{R}_u + \frac{q u^2}{32\pi\varepsilon_0 R^2 c^2}\sin(2\theta)\boldsymbol{\Theta}_u\end{aligned} \tag{29}$$

where $u$ is the linear velocity of the rim of the disc.

*2.3. A small rotating sphere*

Replacing in equation (26) $b^4$ by $(b^2 - y'^2)^2$ and $\tau$ by $\mathrm{d}y'$, and assuming that the disc whose electrokinetic field is represented by equation (26) is a differential element of a sphere, we find by integrating over $y'$ from $-b$ to $b$ that the electrokinetic field of a rotating sphere of radius $b \ll R$ is

$$\boldsymbol{E}_{\mathrm{k}} \approx \frac{\rho\omega^2 b^5}{15\varepsilon_0 R^2 c^2}(\sin^2\theta \boldsymbol{R}_u + \sin\theta\cos\theta \boldsymbol{\Theta}_u). \tag{30}$$

For the Coulomb field we have

$$\boldsymbol{E}_{\mathrm{C}} = \frac{\rho b^3}{3\varepsilon_0 R^2}\boldsymbol{R}_u. \tag{31}$$

Adding equations (30) and (31), we obtain for the total electric field of the rotating sphere

$$\boldsymbol{E} \approx \frac{\rho b^3}{3\varepsilon_0 R^2}\left(1 + \frac{\omega^2 b^2}{5c^2}\sin^2\theta\right)\boldsymbol{R}_u + \frac{\rho\omega^2 b^5}{30\varepsilon_0 R^2 c^2}\sin(2\theta)\boldsymbol{\Theta}_u. \tag{32}$$

Expressing $\boldsymbol{E}$ in terms of the charge of the sphere, $q = 4\pi b^3\rho/3$, we have

$$\begin{aligned}\boldsymbol{E} &\approx \frac{q}{4\pi\varepsilon_0 R^2}\left(1 + \frac{\omega^2 b^2}{5c^2}\sin^2\theta\right)\boldsymbol{R}_u + \frac{q\omega^2 b^2}{40\pi\varepsilon_0 R^2 c^2}\sin(2\theta)\boldsymbol{\Theta}_u \\ &= \frac{q}{4\pi\varepsilon_0 R^2}\left(1 + \frac{u^2}{5c^2}\sin^2\theta\right)\boldsymbol{R}_u + \frac{q u^2}{40\pi\varepsilon_0 R^2 c^2}\sin(2\theta)\boldsymbol{\Theta}_u\end{aligned} \tag{33}$$

where $u$ is the equatorial linear velocity of the sphere.

*2.4. A long rotating hollow cylinder*

Consider a long hollow cylinder of radius $b$ rotating about its symmetry axis. Let the cylinder carry a uniformly distributed charge of density $\rho$ and let the length of the cylinder be $2L$. The electrokinetic field outside the cylinder can be obtained by assuming that the ring whose electrokinetic field is represented by equation (16) is a differential element of the cylinder. Let the thickness of the cylinder wall be $w \ll b$. The electrokinetic field of the cylinder can then be found by replacing in equation (16) $S$ by $w\,\mathrm{d}y'$, $y$ by $y'$, and by integrating the resulting equation over the length of the cylinder.

Let us assume that the point of observation is in the middle plane of the cylinder and that $L \gg x$ ('long' cylinder). We then have

$$\boldsymbol{E}_{\mathrm{k}} = \boldsymbol{i}\frac{\rho u^2 w}{4\pi\varepsilon_0 c^2}\int_0^{2\pi}\int_{-L}^{L}\frac{\cos\varphi}{(x^2 - 2bx\cos\varphi + b^2 + y'^2)^{1/2}}\,\mathrm{d}\varphi\,\mathrm{d}y'. \tag{34}$$

Integrating by parts over $\varphi$, we have

$$\boldsymbol{E}_{\mathrm{k}} = \boldsymbol{i}\frac{\rho u^2 w x b}{4\pi\varepsilon_0 c^2}\int_0^{2\pi}\int_{-L}^{L}\frac{\sin^2\varphi}{(x^2 - 2bx\cos\varphi + b^2 + y'^2)^{3/2}}\,\mathrm{d}\varphi\,\mathrm{d}y'. \tag{35}$$

Integrating over $y'$ and taking into account that $L \gg x$, we obtain

$$\boldsymbol{E}_{\mathrm{k}} = \boldsymbol{i}\frac{\rho u^2 w x b}{2\pi\varepsilon_0 c^2}\int_0^{2\pi}\frac{\sin^2\varphi}{(x^2 - 2bx\cos\varphi + b^2)}\,\mathrm{d}\varphi. \tag{36}$$

The integral in equation (36) is just $\pi/x^2$. The electrokinetic field of the cylinder is therefore

$$\boldsymbol{E}_\mathrm{k} = \frac{\rho u^2 w b}{2\varepsilon_0 x c^2}\boldsymbol{i}. \tag{37}$$

For the Coulomb field we have (see, for example, [7, pp 89–90])

$$\boldsymbol{E}_\mathrm{C} = \frac{\rho w b}{\varepsilon_0 x}\boldsymbol{i}. \tag{38}$$

Adding equations (37) and (38) and using cylindrical coordinates, we obtain

$$\boldsymbol{E} = \frac{\rho w b}{\varepsilon_0 r_0^2}\left(1 + \frac{u^2}{2c^2}\right)\boldsymbol{r}_0 \tag{39}$$

where $\boldsymbol{r}_0$ is a radius vector normal to the axis of the cylinder and directed from the axis to the point of observation; $r_0$ (the magnitude of $\boldsymbol{r}_0$) is the distance from the axis to the point of observation. Expressing $\boldsymbol{E}$ in terms of the charge of the cylinder, $q = 4\pi b w L\rho$, we have

$$\boldsymbol{E} = \frac{q}{4\pi\varepsilon_0 L r_0^2}\left(1 + \frac{u^2}{2c^2}\right)\boldsymbol{r}_0 = \frac{q}{4\pi\varepsilon_0 L r_0^2}\left(1 + \frac{\omega^2 b^2}{2c^2}\right)\boldsymbol{r}_0 \tag{40}$$

where $\omega$ is the angular velocity of the cylinder.

### 2.5. *A long rotating solid cylinder*

Expressing in equation (37) the linear velocity $u$ in terms of the angular velocity $\omega$, replacing $b$ by $x'$ and $w$ by $\mathrm{d}x'$, and considering the hollow cylinder to be a differential element of the solid cylinder, we can write

$$\mathrm{d}\boldsymbol{E}_\mathrm{k} = \boldsymbol{i}\frac{\rho\omega^2 x'^3}{2\varepsilon_0 x c^2}\,\mathrm{d}x'. \tag{41}$$

Integrating equation (41) between 0 and $b$, we obtain for the electrokinetic field of a solid rotating cylinder of radius $b$

$$\boldsymbol{E}_\mathrm{k} = \frac{\rho\omega^2 b^4}{8\varepsilon_0 x c^2}\boldsymbol{i}. \tag{42}$$

For the Coulomb field we have [7, pp 89–90]

$$\boldsymbol{E}_\mathrm{C} = \frac{\rho b^2}{2\varepsilon_0 x}\boldsymbol{i}. \tag{43}$$

Adding equations (42) and (43) and using cylindrical coordinates we obtain

$$\boldsymbol{E} = \frac{\rho b^2}{2\varepsilon_0 r_0^2}\left(1 + \frac{\omega^2 b^2}{4c^2}\right)\boldsymbol{r}_0. \tag{44}$$

Expressing $\boldsymbol{E}$ in terms of the charge of the cylinder, $q = 2\pi b^2 L\rho$, we have

$$\boldsymbol{E} = \frac{q}{4\pi\varepsilon_0 L r_0^2}\left(1 + \frac{\omega^2 b^2}{4c^2}\right)\boldsymbol{r}_0 = \frac{q}{4\pi\varepsilon_0 L r_0^2}\left(1 + \frac{u^2}{4c^2}\right)\boldsymbol{r}_0 \tag{45}$$

where $u$ is the linear velocity at the surface of the cylinder.

*2.6. A long solenoid*

Consider a long solenoid whose symmetry axis is the $y$ axis. Let the length of the solenoid be $2L$ and let it carry a current $I$. The electrokinetic field outside the solenoid can be obtained from equation (37) for the long thin rotating cylinder. Let the thickness of the current-carrying wall of the solenoid be $w$, let the current in each turn of the solenoid be $I$ and let the solenoid have $n$ turns. The charge density $\rho$ associated with the current in the solenoid is then $\rho = nI/2Luw$. Expressing $\boldsymbol{E}_\mathrm{k}$ in terms of $I$, we then obtain

$$\boldsymbol{E}_\mathrm{k} = \frac{nIub}{4\varepsilon_0 Lxc^2}\boldsymbol{i} = \frac{nIub}{4\varepsilon_0 Lr_0^2c^2}\boldsymbol{r}_0 \tag{46}$$

where $\boldsymbol{r}_0$ is a radius vector normal to the axis of the solenoid and directed from the axis to the point of observation; $r_0$ (the magnitude of $\boldsymbol{r}_0$) is the distance from the axis of the solenoid to the point of observation. Since ordinarily only negative charges produce the current in the solenoid, whereas an equal number of positive charges are at rest in the solenoid, the positive charges make no contribution to the electrokinetic field of the solenoid, and the Coulomb fields of the positive and negative charges of the solenoid cancel each other.

## 3. Theory II. Magnetic field

Let us designate the magnetic flux density field associated with the second integral in equation (2) as $\boldsymbol{B}_\mathrm{k}$ and, by analogy with the electrokinetic field $\boldsymbol{E}_\mathrm{k}$, let us call it the 'magnetokinetic field'. We thus have

$$\boldsymbol{B}_\mathrm{k} = \frac{\mu_0}{4\pi c}\int \frac{1}{r^2}\left[\frac{\partial \boldsymbol{J}}{\partial t}\right] \times \boldsymbol{r}\,\mathrm{d}V. \tag{47}$$

To determine the magnetokinetic field produced by the rotating ring shown in figure 1, we proceed as follows. Since at any point of the ring the derivative $\partial\boldsymbol{J}/\partial t$ is the same at all times, the retardation may be assumed to have no effect and we can write equation (47) as

$$\boldsymbol{B}_\mathrm{k} = \frac{\mu_0}{4\pi c}\int \frac{1}{r^2}\frac{\partial \boldsymbol{J}}{\partial t} \times \boldsymbol{r}\,\mathrm{d}V. \tag{48}$$

Just as for the electrokinetic field $\boldsymbol{E}_\mathrm{k}$, $\boldsymbol{r}$ is given by equation (8) and the derivative $\partial\boldsymbol{J}/\partial t$ for the shaded element in figure 1 is given by equations (10)–(12), so that $\mathrm{d}\boldsymbol{B}_\mathrm{k}$ associated with the shaded element is

$$\mathrm{d}\boldsymbol{B}_\mathrm{k} = \frac{\mu_0\rho u^2 \boldsymbol{b}\times\boldsymbol{r}}{4\pi c b^2(x^2 - 2bx\cos\varphi + b^2 + y^2)}\,\mathrm{d}V \tag{49}$$

and, since by equation (7) $\boldsymbol{r} = \boldsymbol{b} + \boldsymbol{R}$, we obtain

$$\mathrm{d}\boldsymbol{B}_\mathrm{k} = \frac{\mu_0\rho u^2 \boldsymbol{b}\times\boldsymbol{R}}{4\pi c b^2(x^2 - 2bx\cos\varphi + b^2 + y^2)}\,\mathrm{d}V \tag{50}$$

which, with equation (6), yields

$$\mathrm{d}\boldsymbol{B}_\mathrm{k} = -\frac{\mu_0\rho u^2(\cos\varphi\boldsymbol{i} + \sin\varphi\boldsymbol{k})\times\boldsymbol{R}}{4\pi c b(x^2 - 2bx\cos\varphi + b^2 + y^2)}\,\mathrm{d}V. \tag{51}$$

The magnetokinetic field produced by the ring is therefore

$$\boldsymbol{B}_\mathrm{k} = \frac{\mu_0\rho u^2}{4\pi c b}\boldsymbol{R}\times\int \frac{\cos\varphi\boldsymbol{i} + \sin\varphi\boldsymbol{k}}{(x^2 - 2bx\cos\varphi + b^2 + y^2)}\,\mathrm{d}V \tag{52}$$

where we have factored out the constant vector $\boldsymbol{R}$. Writing $\mathrm{d}V$ as $Sb\,\mathrm{d}\varphi$, we have

$$\boldsymbol{B}_{\mathrm{k}} = \frac{\mu_0 \rho u^2 S}{4\pi c} \boldsymbol{R} \times \int_0^{2\pi} \frac{\cos\varphi \boldsymbol{i} + \sin\varphi \boldsymbol{k}}{(x^2 - 2bx\cos\varphi + b^2 + y^2)} \,\mathrm{d}\varphi. \tag{53}$$

By symmetry, the contribution of the $z$ component of the integrand is zero, so that we are left with

$$\boldsymbol{B}_{\mathrm{k}} = \boldsymbol{R} \times \boldsymbol{i} \frac{\mu_0 \rho u^2 S}{4\pi c} \int_0^{2\pi} \frac{\cos\varphi}{(x^2 - 2bx\cos\varphi + b^2 + y^2)} \,\mathrm{d}\varphi \tag{54}$$

and, since $\boldsymbol{R} \times \boldsymbol{i} = -\boldsymbol{R}\cos\theta \boldsymbol{k} = -\boldsymbol{y}\boldsymbol{k}$, we obtain

$$\boldsymbol{B}_{\mathrm{k}} = -\boldsymbol{k} \frac{\mu_0 \rho u^2 y S}{4\pi c} \int_0^{2\pi} \frac{\cos\varphi}{(x^2 - 2bx\cos\varphi + b^2 + y^2)} \,\mathrm{d}\varphi. \tag{55}$$

Let us now apply equation (55) to the special cases considered in section 2.

### 3.1. A small rotating ring

Assuming that $b \ll R$ ('small ring'), we have, as in equation (17),

$$1/(x^2 - 2bx\cos\varphi + b^2 + y^2) \approx \frac{1 + 2bx\cos\varphi/R^2}{R^2}. \tag{56}$$

Substituting equation (56) into (55), we obtain

$$\begin{aligned} \boldsymbol{B}_{\mathrm{k}} &\approx -\boldsymbol{k} \frac{\mu_0 \rho u^2 y S}{4\pi c} \int_0^{2\pi} \frac{\cos\varphi(1 + 2bx\cos\varphi/R^2)}{R^2} \,\mathrm{d}\varphi \\ &= -\boldsymbol{k} \frac{\mu_0 \rho u^2 b y x S}{2cR^4} \end{aligned} \tag{57}$$

or

$$\boldsymbol{B}_{\mathrm{k}} \approx -\frac{\mu_0 \rho u^2 b S \sin(2\theta)}{4cR^2} \boldsymbol{k}. \tag{58}$$

In spherical coordinates, $\boldsymbol{B}_{\mathrm{k}}$ is

$$\boldsymbol{B}_{\mathrm{k}} \approx -\frac{\mu_0 \rho u^2 b S \sin(2\theta)}{4cR^2} \boldsymbol{\Phi}_u \tag{59}$$

where $\boldsymbol{\Phi}_u$ is the azimuthal unit vector left-handed relative to the $y$ axis. Written in terms of the current $I = \rho u S$ created by the rotating ring, $\boldsymbol{B}_{\mathrm{k}}$ is

$$\boldsymbol{B}_{\mathrm{k}} \approx -\frac{\mu_0 I u b \sin(2\theta)}{4cR^2} \boldsymbol{\Phi}_u. \tag{60}$$

The total magnetic field produced by the ring is the sum of $\boldsymbol{B}_{\mathrm{k}}$ and the field associated with the first integral in equation (2). However, the latter field (which is the ordinary dipole field of the ring) is of no interest for the present discussion and we shall not include it in our derivations.

### 3.2. A small rotating disc

Expressing in equation (59) the linear velocity $u$ in terms of the angular velocity $\omega$, replacing $b$ by $x'$ and $S$ by $\tau\,\mathrm{d}x'$, and considering the ring to be a differential element of a disc, we can write

$$\mathrm{d}\boldsymbol{B}_{\mathrm{k}} \approx -\boldsymbol{\Phi}_u \frac{\mu_0 \rho \omega^2 x'^3 \tau \sin(2\theta)}{4cR^2} \,\mathrm{d}x'. \tag{61}$$

Integrating equation (61) between 0 and $b$, we then obtain for the magnetokinetic field of a disc of radius $b \ll R$

$$\boldsymbol{B}_{\mathrm{k}} \approx -\frac{\mu_0 \rho \omega^2 b^4 \tau}{16 c R^2} \sin(2\theta) \boldsymbol{\Phi}_u = -\frac{\mu_0 \rho u^2 b^2 \tau}{16 c R^2} \sin(2\theta) \boldsymbol{\Phi}_u \tag{62}$$

where $u$ is the linear velocity of the rim of the disc. Expressing $\boldsymbol{B}_{\mathrm{k}}$ in terms of the charge of the disc, $q = \pi b^2 \tau \rho$, we obtain

$$\boldsymbol{B}_{\mathrm{k}} \approx -\frac{\mu_0 q \omega^2 b^2}{16 \pi c R^2} \sin(2\theta) \boldsymbol{\Phi}_u = -\frac{\mu_0 q u^2}{16 \pi c R^2} \sin(2\theta) \boldsymbol{\Phi}_u. \tag{63}$$

*3.3. A small rotating sphere*

Using equation (62) with angular velocity $\omega$, replacing $b^4$ by $(b^2 - y'^2)^2$ and $\tau$ by $\mathrm{d}y'$, and assuming that the disc represented by equation (62) is a differential element of a sphere, we find by integrating from $-b$ to $b$ that the magnetokinetic field of the rotating sphere of radius $b \ll R$ is

$$\boldsymbol{B}_{\mathrm{k}} \approx -\frac{\mu_0 \rho \omega^2 b^5 \sin(2\theta)}{15 c R^2} \boldsymbol{\Phi}_u = -\frac{\mu_0 \rho u^2 b^3 \sin(2\theta)}{15 c R^2} \boldsymbol{\Phi}_u \tag{64}$$

where $u$ is the equatorial linear velocity of the sphere. Expressing $\boldsymbol{B}_{\mathrm{k}}$ in terms of the charge of the sphere, $q = 4\pi b^3 \rho / 3$, we obtain

$$\boldsymbol{B}_{\mathrm{k}} \approx -\frac{\mu_0 q \omega^2 b^2 \sin(2\theta)}{20 \pi c R^2} \boldsymbol{\Phi}_u = -\frac{\mu_0 q u^2 \sin(2\theta)}{20 \pi c R^2} \boldsymbol{\Phi}_u. \tag{65}$$

*3.4. Long rotating cylinders and a long solenoid*

Consider a long thin rotating cylinder whose symmetry axis is the $y$ axis. Let the length of the cylinder be $2L$, let the thickness of its wall be $w$, and let it carry a uniformly distributed charge of density $\rho$. The magnetokinetic field outside the cylinder can be obtained with the help of equation (55) by assuming that the ring, whose electrokinetic field is represented by equation (55), is a differential element of the cylinder. Let the point of observation be in the middle plane of the cylinder and let $L \gg x$ ('long' cylinder). Replacing $y$ in equation (55) by $y'$, and replacing $S$ by $w\,\mathrm{d}y'$, we then have

$$\begin{aligned}\boldsymbol{B}_{\mathrm{k}} &= -\boldsymbol{k} \int_{-L}^{L} \frac{\mu_0 \rho u^2 w y'}{4\pi c} \int_0^{2\pi} \frac{\cos\varphi}{(x^2 - 2bx\cos\varphi + b^2 + y'^2)} \,\mathrm{d}\varphi\,\mathrm{d}y' \\ &= -\boldsymbol{k} \frac{\mu_0 \rho u^2 w}{4\pi c} \int_{-L}^{L} \int_0^{2\pi} \frac{y' \cos\varphi}{(x^2 - 2bx\cos\varphi + b^2 + y'^2)} \,\mathrm{d}\varphi\,\mathrm{d}y'. \end{aligned} \tag{66}$$

However, the contributions of $-y'$ and $y'$ cancel when we integrate from $-L$ to $L$. Thus the integral in equation (66) is zero and the cylinder does not create a magnetokinetic field in the midplane of the cylinder. The same considerations apply to a solid rotating cylinder and to a current-carrying solenoid.

## 4. Discussion

The calculations of the electrokinetic and magnetokinetic fields $\boldsymbol{E}_{\mathrm{k}}$ and $\boldsymbol{B}_{\mathrm{k}}$ presented above reveal the existence of several previously unreported and unforeseen electromagnetic effects, the most important of which are:

(1) A steady-state circular electric current creates a constant radial electric field associated with radial acceleration of the charges constituting the current.

(2) A uniformly rotating spherical charge creates not only the usual Coulomb field, but also a constant radial electric field associated with radial acceleration of the elementary charges within the spherical charge under consideration.

(3) A steady-state circular electric current creates not only the usual magnetic field associated with the current, but also a constant circular magnetic field associated with radial acceleration of the charges constituting the current.

(4) A uniformly rotating spherical charge creates not only the usual magnetic dipole field, but also a constant circular magnetic field associated with radial acceleration of the elementary charges within the spherical charge under consideration.

In connection with these effects two questions arise:

(1) Why were these effects not predicted by electromagnetic theory in the past?

(2) Are there any experiments manifesting these effects?

The answer to the first question is quite simple. Crucial for the calculations presented in this paper are equations (1) and (2). Although these equations were derived almost half a century ago, they became generally known relatively recently and have not yet been used for practical applications to any appreciable extent.

The answer to the second question is more complicated. First, it should be noted that, because of the factor $u^2/c^2$ in the electrokinetic field equations and the factor $u^2/c$ in the magnetokinetic field equations, these fields are extremely weak, except when $u$ is close to $c$, which can hardly happen in macroscopic systems. Furthermore, in order to detect these very weak fields in macroscopic experiments one needs to use field-detecting devices whose sensitivity is much greater than that of conventional macroscopic electromagnetic instruments.

The situation is quite different in mesoscopic and microscopic systems. High velocities are commonplace in such systems, and electric and magnetic fields are detected there by elementary particles whose motion is strongly affected even by extremely weak fields. In fact, there is good evidence that the electrokinetic field associated with a circular current has already been observed in a mesoscopic experiment, in the well known Aharonov–Bohm experiment.

In their now famous 1959 paper, Aharonov and Bohm advanced the theory that the magnetic vector potential is not just a mathematical device, but an observable physical quantity having a physical significance of its own [11]. They suggested that the wavefunction of electrons moving outside a long current-carrying solenoid in a region where (as was generally assumed) there are no electric or magnetic fields can be altered solely by the magnetic vector potential. Experiments have apparently supported their theory [12], and the phenomenon predicted by them became known as the Aharonov–Bohm effect.

Although the Aharonov–Bohm effect is considered to be a quantum mechanical effect (it manifests itself as a shift of wavefunction interference fringes), it could have profound and dramatic consequences for classical electrodynamics. In classical electrodynamics, only electric and magnetic fields can alter the motion of charged particles. They do so by exerting electric and magnetic forces on the particles. If electromagnetic potentials can exert a dynamical effect on charged particles in the absence of electric or magnetic fields at the location of the particles, then the entire Maxwellian electrodynamics is incorrect and must be replaced by a different theory. However, no physical theory has been validated more convincingly and more completely than Maxwellian electrodynamics.

It should be noted that the expressions for the fringe shift obtained by Aharonov and Bohm involve purely classical electrodynamic quantities: the macroscopic magnetic vector potential of the solenoid and the macroscopic magnetic flux inside the solenoid in particular. The question arises therefore: is it possible that the fringe shift observed in the Aharonov–Bohm experiment is actually associated with an electric or magnetic field outside the solenoid that somehow was overlooked in the formulation of the theory of the effect? Numerous attempts have been made to interpret this fringe shift in terms of classical electrodynamics [13], and the role of the magnetic vector potential in the Aharonov–Bohm experiment has been questioned [14]. Quite clearly, a successful electrodynamic theory of the Aharonov–Bohm experiment should accomplish two things: it should show that an electric or magnetic field does indeed exist outside the solenoid in the Aharonov–Bohm experiment and it should explain the role of the macroscopic magnetic vector potential in that experiment. Until now such a theory could not be formulated.

The theory of acceleration-related electrokinetic fields presented in this paper finally makes it possible to reconcile the Aharonov–Bohm experiment with classical electrodynamics. According to equation (46), there is indeed an electric (electrokinetic) field outside a current-carrying solenoid:

$$\boldsymbol{E}_{\rm k} = \frac{nIub}{4\varepsilon_0 L r_0^2 c^2}\boldsymbol{r}_0. \tag{46}$$

Therefore, according to this equation, the electrons in the Aharonov–Bohm experiment are moving in the presence of an electric field and, consequently, experience an electric force due to this field. The question remains, however, how this electric force is related to the magnetic vector potential of the solenoid. The answer to this question is provided below.

As with any electric field, the electrokinetic field exerts a force

$$\boldsymbol{F} = q\boldsymbol{E}_{\rm k} \tag{67}$$

on the electric charge $q$ located in this field. If the charge $q$ moves through $\boldsymbol{E}_{\rm k}$, it acquires a momentum $\boldsymbol{p}$ given by

$$\boldsymbol{p} = \int \boldsymbol{F}\,{\rm d}t = \int q\boldsymbol{E}_{\rm k}\,{\rm d}t. \tag{68}$$

Substituting $\boldsymbol{E}_{\rm k}$ from equation (4) and integrating over time, we have

$$\boldsymbol{p} = -\frac{q}{4\pi\varepsilon_0 c^2}\iint \frac{1}{r}\frac{\partial \boldsymbol{J}}{\partial t}\,{\rm d}V\,{\rm d}t = -\frac{q}{4\pi\varepsilon_0 c^2}\int \frac{\boldsymbol{J}}{r}\,{\rm d}V. \tag{69}$$

However, the last integral in equation (69) is

$$\frac{1}{4\pi\varepsilon_0 c^2}\int \frac{\boldsymbol{J}}{r}\,{\rm d}V = \boldsymbol{A} \tag{70}$$

where $\boldsymbol{A}$ is the vector potential produced at any point of space by the current $\boldsymbol{J}$ (see, for example, [7, pp 363–4] noting that $1/\varepsilon_0 c^2 = \mu_0$), the current which, according to equation (2), also produces the magnetic field $\boldsymbol{B}$ in the solenoid. Thus, for an electron in the Aharonov–Bohm experiment, equation (69) can be written as

$$\boldsymbol{p} = e\boldsymbol{A} \tag{71}$$

where $\boldsymbol{A}$ is the vector potential at the point where the electron (*negative* charge $e$) is located.

When the electron moves from its source to the screen on which the interference fringes are observed, the phase of its wavefunction changes by [15]

$$\delta = \int \frac{\boldsymbol{p}}{\hbar}\cdot{\rm d}\boldsymbol{l} = \frac{e}{\hbar}\int \boldsymbol{A}\cdot{\rm d}\boldsymbol{l}. \tag{72}$$

The difference in the phases of the wavefunction of electrons passing the solenoid on two different sides $S_1$ and $S_2$ is then

$$\delta_1 - \delta_2 = \frac{e}{\hbar}\left(\int_{S_1}\boldsymbol{A}\cdot{\rm d}\boldsymbol{l} - \int_{S_2}\boldsymbol{A}\cdot{\rm d}\boldsymbol{l}\right) = \frac{e}{\hbar}\oint \boldsymbol{A}\cdot{\rm d}\boldsymbol{l}. \tag{73}$$

Since

$$\oint \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{l} = \Phi \tag{74}$$

where $\Phi$ is the magnetic flux enclosed by the path of integration (see, for example, [7, p 366]) (the magnetic flux inside the solenoid), we finally obtain

$$\delta_1 - \delta_2 = \frac{e}{\hbar}\Phi. \tag{75}$$

This is the same equation as that obtained by Aharonov and Bohm on the basis of their theory.

Thus, equation (75), which is the mathematical expression for phase shift in the Aharonov–Bohm experiment, is a consequence of an electric force acting on moving electrons. The force is caused by the electric field produced by the same current which creates the magnetic field in the solenoid. This force is a strictly classical phenomenon and is a direct consequence of Maxwell's electrodynamic equations. Therefore, although Aharonov and Bohm attributed the phase shift observed in their experiment to the magnetic vector potential, the experiment does not reveal any special physical significance of the magnetic vector potential[2]. In this connection it may be mentioned that electromagnetic forces can be quite generally described and computed not only in terms of electromagnetic fields but also in terms of electromagnetic potentials [17].

In conclusion it should be noted that the study of the electrokinetic and magnetokinetic fields represented by the second integral of equation (1) and by the second integral of equation (2) is still in its rudimentary stage. More studies, especially experimental, are needed for elucidating properties, peculiarities and possible uses of these fields. The calculations presented in this paper indicate that these fields may give rise to a variety of important electromagnetic effects not yet described or observed.

## References


[1] Griffiths D J and Heald M A 1991 Time-dependent generalization of the Biot–Savart and Coulomb laws *Am. J. Phys.* **59** 111–7

[2] Tran-Cong Ton 1991 On the time-dependent, generalized Coulomb and Biot–Savart laws *Am. J. Phys.* **59** 520–8

[3] Heald M A and Marion J B 1995 *Classical Electromagnetic Radiation* (Philadelphia, PA: Saunders) pp 261-2

[4] Rosser W G V 1997 *Interpretation of Classical Electromagnetism* (Dordrecht: Kluwer) pp 48–9
Rosser W G V 1997 *Interpretation of Classical Electromagnetism* (Dordrecht: Kluwer) pp 82–4

[5] Jackson J D 1999 *Classical Electrodynamics* 3rd edn (New York: Wiley) pp 246–7

[6] Griffiths D J 1999 *Introduction to Electrodynamics* 3rd edn (Englewood Cliffs, NJ: Prentice-Hall) pp 427–8

[7] Jefimenko O D 1992 *Electricity and Magnetism* 2nd edn (Star City: Electret Scientific) (1st edn 1966 New York: Appleton-Century-Crofts)

[8] Jefimenko O D 2000 *Causality, Electromagnetic Induction, and Gravitation* 2nd edn (Star City: Electret Scientific) pp 28–30

[9] Jefimenko O D 1994 Direct calculation of the electric and magnetic fields of an electric point charge moving with constant velocity *Am. J. Phys.* **62** 79–85

[10] Jefimenko O D 1997 *Electromagnetic Retardation and Theory of Relativity* (Star City: Electret Scientific) pp 62–70

[11] Aharonov Y and Bohm D 1959 Significance of electromagnetic potentials in the quantum theory *Phys. Rev.* **115** 485–91


[2] Several variations of the original Aharonov–Bohm experiment have been reported. In all probability the fringe shift observed in such experiments is caused by a combination of several different factors. Depending on the experimental arrangement, one or another of these factors may predominate. An experiment that has attracted much attention involved a ferromagnet enclosed in a superconducting shield [16]. It is quite clear that in this particular experiment the electrons passing the shielded magnet created electric images in the superconducting shield (see, e.g., [7] pp 161–8). These images inevitably created an electric image field around the shield, so that, contrary to the supposition, the electrons were not moving in a field-free region at all.

[12] See, for example, Peshkin M and Tonomura A 1989 *The Aharonov–Bohm Effect* (New York: Springer)
[13] For a discussion, see Portis A M 1978 *Electromagnetic Fields: Sources and Media* (New York: Wiley) pp 400–5 and references cited there
[14] See, for example, Boyer T H 2000 Does the Aharonov–Bohm effect exist *Found. Phys.* **30** 893–905
Boyer T H 2000 Classical electromagnetism and the Aharonov–Bohm phase shift *Found. Phys.* **30** 907–32 and papers cited there
[15] See, for example, Feynman R P, Leighton R B and Sands M 1964 *The Feynman Lectures on Physics* vol 2 (Reading, MA: Addison-Wesley) section 15 p 9
[16] Osakabe N, Matsuda T, Kawasaki T, Endo J, Tonomura A, Yano S and Yamada 1986 Experimental confirmation of the Aharonov–Bohm effect using a toroidal magnetic field confined by a superconductor *Phys. Rev.* A **34** 815–22
[17] Jefimenko O D 1990 Direct calculation of electric and magnetic forces from potentials *Am. J. Phys.* **58** 625–31